The belt trick , easy with Geometric Algebra.

The belt trick is attributed to P.A.M.Dirac ; he is supposed to have invented it in order to help his students struggling with the subtleties of spinors in quantum mechanics. Alas nobody seems to have noted his detailed explanations, and I must confess that I have until now not found on Internet a completely satisfying text, both convincing and not relying on abstract theoretical mathematics. So I tried to find myself something new, both in an informal mode and also tentatively in a rigorous mathematical approach, with geometric algebra. I hope having succeeded ; if not I expect helpful critics.

1/ A surprisingly easy general demonstration almost without math.

Of course the first approach of the belt trick should be an experimental one. Let us consider a belt[1] disposed flat on a table, with the belt buckle situated near the operator (south) and the open end situated north. One rotates twice – a 4π turn – the belt buckle around the south-north axis, then translates the buckle on the table northward without changing its orientation (a parallel move), and, exploiting the souple middle part of the belt, moves that part around the buckle by passing under it. Then straightening again the belt (translation of the buckle) one sees that it is miraculously untwisted ! Finally doing again the same operation, after a 2π only turn of the buckle one sees that it remains twisted.

It is not easy at all to analyse the move of the middle part of the belt ; one may have the impression of some magician trick ! Luckily it is very easy to verify experimentally that the topological conditions are not changed if we adopt the following alternative procedure :

– First, we move the buckle in a translational parallel transport on the table, completely around some arbitrary point chosen on the middle part of the belt (that is an horizontal global rotation of the buckle without ever changing its intrinsic orientation on the table), and passing over it[2]. Let us make a temporary stop at the precise moment when the buckle overcrosses the northern part of the belt. We observe that the silver thread and the gold thread are correctly paired. That means that we already have twisted the south part with a 2π rotation (around a south-north axis)[3]. If we now finish our parallel rotational move, restoring the buckle in its initial position, we observe that we have twisted the belt with a 4π rotation (around a south-north axis), as the silver and gold threads remain correctly paired.

– Second, starting again with a flat belt we do the inverse moves , first twisting the belt twice (4π) around the south-north axis, then doing the translational parallel motion of the buckle on the table. Obviously there are two possibilities ; if we choose the right[4] one we untwist completely the belt, alternatively we finish with a 4-twisted belt (8π) !

– Finally let us begin with a simple (2π) twisting of the belt and try the untwisting parallel translation move of the buckle. The result is obvious now : we finish either with a simple twist (in the opposite direction) or with a triple twist (6π) !

The conclusion is : even twisted belts remain even, which includes the flat belt, and odd twisted ones remain odd.

All our experimentation is done in $\mathcal{R}^3$, a real space ; can we find a link with the abstract complex $SU(2)$ space ? That remains for me an open question.

1. To facilitate the observation and description we suppose that the belt is bordered on the westside by a golden thread, and on the eastside by a silver thread.

2. In more detail : pinning down the belt approximately in the middle with a left hand finger you rotate completely the buckle around that finger for example from east to west, overcrossing the northern half of the belt, and finally extending again the belt southwards, while never changing the horizontal orientation of the buckle. Not easy to explain but easy to do !

3. Of course at that moment the south part of the belt is upright over the table like a cobra ... (!)

4. You will easily find out which one it is ...

2/How does that look with some not too sophisticated mathematics ?

On the Internet I have found mathematical interpretations with classical quaternions. They seemed to me unnecessary complicated. I think geometric algebra is much more supple and easier to learn and interprete.

Well it is not necessary to utilize that propriety but I cannot resist to draw attention to the fact that a narrow belt made in supple but not too extensible tissue is a developpable surface, which means that there exists on it at least one family of straight lines, which are also geodesics and which subsist when we twist it reasonnably. I mean of course all the infinitesimally adjacent short transverse lines. That inspires me the idea to study a belt made of a great number of parallel needles loosely attached together, in order to permit some relative rotations between them. Then we can try to represent the belt not by the coordinates of each needle but by the small rotation necessary to pass from one to the other. Those rotations will add together in some manner along the flexible south-north axis and thus explain the global rotation between the first and the last needle, which orientations are given in advance.

And it happens that geometric algebra is particularly well adapted to study rotations, in a manner which here gives a stunningly simple and useful formula.

First we must refresh our knowledge in GA. If you look for references see Hestenes [1] where you will find also useful figures. We can represent any θ rotation by an operator $R(\theta)$ called a rotor, which acts double-sidedly on any vector u by $v = Ru\tilde{R}$. R can be given in different representations. Here the most useful is $R = ab$ where a, b are two unit vectors separated by an angle[5] $(\theta/2)$. Then we have the reverse $\tilde{R} = ba$. The fact that the operators can be represented by simple geometric products is essential for what follows.

How shall we compose two successive rotations R_θ , R_φ ? Simply by writing :

(1) $w = R_\varphi R_\theta u \tilde{R}_\theta \tilde{R}_\varphi \qquad R_{\theta,\varphi} = R_\varphi R_\theta \qquad \tilde{R}_{\theta,\varphi} = \tilde{R}_\theta \tilde{R}_\varphi$

The plane (dual of the rotation axis) of the ab rotation is precisely defined by the two unit vectors. If we compose two successive rotations we are free to chose a common intermediate vector b . Thus we write :

(2) $(ab)(bc) = (ac)$

On a unit sphere – see [1] – these rotors are symbolized by arcs of great circles with values[6]

$\theta/2, \varphi/2, \psi/2$!

We are ready now for action. We start by rotating the buckle of the belt by an angle 2π around the south-north axis ; thus we have created a 1-twist. The trajectory of the belt is rather undetermined in $\mathcal{R}^3$ but theoretically we could measure the rotation angle between adjacent needles (!). Then we could represent on the above mentioned 2-dimensional sphere the successive rotations by small arcs forming a continuous line between a point *symbolizing* the buckle and its antipodal point *symbolising* the loose end of the belt. That could be mathematically represented by the formula :

(3) $(a_0 a_1)(a_1 a_2)(a_2 a_3)\ldots\ldots\ldots\ldots\ldots(a_{n-2} a_{n-1})(a_{n-1} a_N) = a_0 a_N = -1$

Why (-1) and thus why antipodal points ? Of course because we must draw on the rotation sphere not the physical θ angles, but the $\theta/2$ angles characterizing the different elementary rotors[7]. They act double-sidedly !

5. Please be careful ; physicists speak of $\mathcal{R}(\theta)$ operators,implicitly refering to matrices, but when they act double-sided as rotors (spinors) each side has a $\theta/2$ parameter.

6. One might understand now why I protest against the curious habit of physicists to proclaim that the spinor (ac) is equal to the spinor (bc) rotated by an angle θ (yes not $\theta/2$) under the action of the spinor (ab) – see [2],[3]. Thus they are lead to present as a mysterious quantum mechanical fact the change of sign of a spinor, which is supposed erroneously to be rotated by a (2π) angle.

7. By the way that physical reality disappears when operating with standard Clifford algebra in abstract SU(2) spaces. That explains perhaps the misunderstandings.

But now there is no way in $\mathcal{R}^3$ to untwist the belt, because that would need on the rotation sphere to put the initial and final points together, which by definition of the rotors and our twisting hypothesis we are not able to do ! They remain separated by an angle π , which corresponds to the (-1) value in relation (3), whatever admissible[8] changes we introduce in the chain of a_i vectors.

On the contrary if we choose an initial twisting by an angle 4π , the relation (3) becomes :

(4) $\quad (a_0\, a_1)(a_1 a_2)(a_2 a_3)..............(a_{n-2} a_{n-1})(a_{n-1} a_N) = a_0\, a_N = 1$

That means of course that the point a_N is coincident with a_0 ! The curve representing step by step the elementary rotations forms a loop which passes also by the antipodal point (or very near). Starting there and modifying step by step the physical belt trajectory we can annihilate progressively the rotation trajectory and finaly completely contract the loop. That means untwisting of the belt. Of course we cannot hope to describe analytically the detailed moves of the belt in $\mathcal{R}^3$, but that is not necessary for the demonstration.

Thus the geometric algebra tool easily enables us to confirm rigorously the experimental results, with a minimum of mathematical apparatus. This is due principally to the suppleness and the great adaptability of that method. As a comparison with a more classical method one can take a look at [4], which necessitates a good knowledge of topologic methods.

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[1] Hestenes Oerstedt Medal Lecture 2002

[2] G.Ringeisen Spineurs ... encore 2010 2013

[3] G.Ringeisen Clifford Algebra or Geometric Algebra 2013

[4] http://gregegan.customer.netspace.net.au/APPLETS/21/DiracNotes.html

8. That is compatible with the belt tissue ...