Clifford algebra or geometric algebra ?

As I began some years ago to study seriously the hestenian geometric algebra I quickly discovered some disturbing facts.

I knew nothing about standard Clifford algebra, and being much more interested in physics than in pure mathematics I would never have dared to open a book dealing with that abstract algebraic speciality. Today I still realize that to understand it in depth I would need several semesters of Bourbakist *reconditioning*, which of course I will never do.

But geometric algebra opened to me new horizons. I immediately found myself in agreement with Hestenes vision of mathematics , constructing it and teaching it. And quickly I realized that his methods where mostly rejected by traditionalists, that the merit of having rejuvenated Clifford algebra whas contested to him, both by jealous professionals and ignorant amateurs (see Wikipedia), and that introduction of GA in the learning cycle was far from a won struggle.

Roughly and perhaps with some caricatural deviations, I would classify the attitudes of professionals and some more or less enlighted amateurs in a few categories :

1/ First of all the theoretical oriented professional algebrists ; they are not at all interested in Hestenes educational ideas and thus consider the GA as non existant. Their work, whose utility I of course do not contest, is at high abstract levels and thus unattainable to amateurs and even perhaps to a majority of professional physicists.

2/The quantum physicists able to manipulate the standard Clifford algebra. They cling of course to the enormous theoretical corpus elaborated during the last hundred years, which undoubtedly has produced splendid results in the understanding of reality. They mostly do not like GA, even reject it violently, not only because they do not want to rewrite their theories, but also because doing quantum physics with GA ineluctably leads to question the prevailing QM interpretations !

3/The *Working Class* in Clifford algebra and/or quantum mechanics, high level students, professors, research people, some amateurs, to whom Hestenes work has brought a new impetus, new ideas which they try to utilize in their work. But they always try to rewrite , justify, the new methods in the standard Clifford algebra. Thus not only they mostly refuse to Hestenes his merits, but also they destroy voluntarily or not the geometrical and synthetic specificities of the GA.

4/A very small minority of professionals and amateurs who have understood, or at least think so, the deep novelty of GA and the ease of its learning process , which authorizes shortcuts through many mathematical difficulties. Let us cite : the UK Cambridge people (those who still work in that field ?), some German professors and students, in Amsterdam Dorst and his students, very few people in the US, Hitzer in Japan, in France a few oldies (I have the privilege to know and exchange ideas with Roger Boudet, an unconventional physicist adept of the de Broglie school, who worked with Hestenes), in France also a few specialists of ray-tracing (University of Poitiers).

I had recently the opportunity to study some parts of Peter Lounesto's book "Clifford algebras and spinors" as an introduction to traditional Clifford algebra, but I was not too happy with it. I will try to give some details later, but let me first draw attention on a remarkable article written by a young German physicist, Florian Jung, " Geometrische Algebra und die Rolle des Clifford-Produkts in der Klassischen und Quantenmechanik ", which you can find on Internet. Lounesto figures in the references, but Jung's text is much more explicit, and thus reader-friendly. The detailed equations are easy to follow even with a limited knowledge of german language. It is probably the best and most complete text I have read on that subject. Jung's article respects the true spirit of GA, and explains the relationship with standard Clifford calculus without giving the impression that Hestenes[1] has " done nothing than giving a snazzy new name to something that already existed for a long time ... ".

1. I read that on Internet . If it already existed why wasn't it exploited in Hestenes manner ?

P-S or P-S-H ?

If you take a look at pages 50 - 64 of Lounesto's book you will probably, unless you are already a specialist of classic Clifford algebra and group theory, find it rather confusing, whether it speaks about quantum mechanics (the justification of the Pauli Schroedinger equation) or about the back and forth switching between $\mathcal{C}l_3$, Mat$(2,\mathcal{C})$, S . Thus rather than trying to explain it and perhaps critizise it too severely, I will start with what I wrote in my Internet article " Spineurs ... encore " and try to show how it can be rendered more rigorous with very simple means, that is without introducing more theoretical notions than simple matrix calculations and of course some knowledge of geometric algebra and elementary quantum mechanics.

I struggled a long time before understanding that the main difficulty about establishing a bijective relation between the standard Pauli - Schroedinger equation (P-S) :

(1) $\quad i\hbar\partial_t\,\Psi = \mathcal{H}_S\,\Psi - \frac{e\hbar}{2mc}\boldsymbol{\sigma}.B\,\Psi$ $\quad$ (where Ψ is a standard column spinor $(\psi_1,\psi_2)^t$)

and the Pauli - Schroedinger - Hestenes (P-S-H) equation :

(2) $\quad \hbar\partial_t\,\phi\,\boldsymbol{e_1}\,\boldsymbol{e_2} = \mathcal{H}_S\,\phi - \frac{e\hbar}{2mc}B_k\,\boldsymbol{e_k}\,\phi\,\boldsymbol{e_3}$ $\quad$ (where ϕ is an even GA multivector)

was not how to switch between (1) and (2) but how to establish an undisputable bijective relation between (1) and (3) where an adequate matrix spinor ψ (or should we say spinor operator ?) is substituted to the column spinor Ψ :

(3) $\quad \hbar\partial_t\,\psi\,\sigma_1\,\sigma_2 = \mathcal{H}_S\,\psi - \frac{e\hbar}{2mc}B_k\,\sigma_k\,\psi\sigma_3$

In that task reading F.Jung helped me a lot. Of course once you have (3) the bijective switch between (2) and (3) is obvious.

But how can we guess what an adequate ψ matrix should be ? I cannot resist to recall the sarcastic remark which I read for the first time in R.Boudet, " La théorie intrinsèque de la particule de Dirac et l'Ecole Louis de Broglie ", noting that the spinor Ψ is nothing else than a disarticulated quaternion (even multivector in $\mathcal{G}_3$) :

(4) $$\Psi = \begin{pmatrix} a_0 + i\,a_3 \\ i\,a_1 - a_2 \end{pmatrix}$$

Then if we translate the multivector $\phi = a_0 + a_k\,I e_k$ in matrices $\psi = a_0\,\mathbf{1} + a_k\,i\sigma_k$ we get :

(5) $$\psi = \begin{pmatrix} a_0 + i\,a_3 & i\,a_1 + a_2 \\ i\,a_1 - a_2 & a_0 - i\,a_3 \end{pmatrix}$$

and simultaneously we note that every even matrix in Mat$(2,\mathbb{C})$ has the structure :

(6) $$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$ (where $\bar{\psi}$ is the complex conjugate of ψ)

We know of course that those matrices constitute a subalgebra of Mat$(2,\mathbb{C})$.

The fact that ψ depends only on 4 parameters instead of the 8 theoretically possible, added to the precisely defined structure of the matrix, is a necessary but not sufficient condition for establishing a bijective relation between (1) and (3). Now we have simply to calculate the 4 differential equations corresponding to the 4 components of the matrices on both sides of (3) and verify that they reduce to two of them. First we establish the $\psi\sigma_3$ and $B_k\,\sigma_k$ matrices :

(7) $$\psi\sigma_3 = \begin{pmatrix} \psi_1 & \bar{\psi}_2 \\ \psi_2 & -\bar{\psi}_1 \end{pmatrix}$$

(8) $$B_k\,\sigma_k = \begin{pmatrix} B_3 & B_1 - i\,B_2 \\ B_1 + i\,B_2 & -B_3 \end{pmatrix}$$

Then we multiply everything with the first column of the ψ and $\psi\sigma_3$ matrices, which gives following results :

$$\hbar i \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mathcal{H}_S \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \frac{e\hbar}{2mc} \begin{pmatrix} B_3\psi_1 + (B_1 - i B_2)\psi_2 \\ (B_1 + i B_2)\psi_1 - B_3\psi_2 \end{pmatrix} \tag{9}$$

The same calculation with the second column gives :

$$\hbar i \partial_t \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix} = \mathcal{H}_S \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix} - \frac{e\hbar}{2mc} \begin{pmatrix} B_3\bar{\psi}_2 - (B_1 - i B_2)\bar{\psi}_1 \\ (B_1 + i B_2)\bar{\psi}_2 + B_3\bar{\psi}_1 \end{pmatrix} \tag{10}$$

which after a complex conjugation gives :

$$\hbar i \partial_t \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} = \mathcal{H}_S \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} - \frac{e\hbar}{2mc} \begin{pmatrix} B_3\psi_2 - (B_1 + i B_2)\psi_1 \\ (B_1 - i B_2)\psi_2 + B_3\psi_1 \end{pmatrix} \tag{11}$$

Thus as equations (11) identify with equations (9) we have demonstrated the bijective relation between (1) and (3) and consequently the bijective relation between the classic P-S equation in QM and the equivalent P-S-H equation.

Dirac equation and Lorentz groups.

With the same methods one can prove the equivalence of the standard Dirac equation and the Dirac-Hestenes equation in quantum mechanics. It is easy to see that in the Dirac (4,4) matrices all columns are deduced from the first one by simple linear transforms and thus that there are not 32 independant parameters (!), but only 8, that is the same number as in 4-dimensional geometric algebra. That presumes a bijective relation.

Now what about the spinors in $\mathcal{C}l(1,3)$? I only studied the unitary spinors (rotors) operating in the orthochrone proper Lorentz group. I wrote an article which figures on my Internet site, in French, but with more equations than text : " La transformation de Lorentz vue en algèbre géométrique ". I hope having correctly established the following conclusions :

– The rotor R can be written as the product, not uniquely, of a spatial rotation U and a boost L , that is :

$$R = LU \qquad\qquad f = R e \tilde{R} = LUe\tilde{U}\tilde{L} \tag{12}$$

– If we guess that the complete rotor can be written :

$$R = \alpha + \boldsymbol{a} + i\,\boldsymbol{b} + i\beta \qquad \text{with} \qquad \boldsymbol{a} = a^k\gamma_k\gamma_0 \qquad \boldsymbol{b} = b^k\gamma_k\gamma_0 \qquad R\tilde{R} = 1 \tag{13}$$

we find that we must satisfy the following not so obvious conditions :

$$\beta = 0 \qquad \boldsymbol{a}.\boldsymbol{b} = 0 \qquad \alpha^2 - \beta^2 - \boldsymbol{a}^2 + \boldsymbol{b}^2 = 1 \tag{14}$$

If we replace $\beta = 0$ by $\alpha = 0$ the resulting rotor will realize a space inversion without time inversion.

– The relation between initial basis, final basis and the rotor is :

$$R = f_\mu e^\mu / 4\alpha = f_\mu e^\mu / (f_\mu e^\mu e^\nu f_\nu)^{1/2} \tag{15}$$

Thus the idea of spinors being defined in GA by even multivectors could possibly be generalized, with some restrictions related to the tasks we want them to do. So it is not surprising to find different opinions and definitions between high level mathematicians, specialized quantum theoricians, laboratory physicists, some amateurs, etc ...

A more general demonstration.

Inspired by a very intersting dissertation by Shyamal S. Somaroo, " Applications of the Geometric Algebra to Relativistic Quantum Theory ", I try to rewrite a more elegant and especially more general demonstration – it must already exist in some paper – , for the switch between equations (1) and (3) . The general idea is to show, while staying in the even subspace of Math$(2, \mathcal{C})$ that we can multiply (3) not only by the column vector $u = (1,0)^T$, but also by $u' = (0,\ 1)^T$ and thus by any linear combination of both of them. Thus we can simplify (3) by u which can be immediately rewritten in GA as (2).

Let us start again with :

(16) $$i\hbar\partial_t \Psi = \mathcal{H}_S \Psi - \frac{e\hbar}{2mc} \boldsymbol{\sigma}.B\Psi$$

which we transform in :

(17) $$\hbar\partial_t \psi \sigma_1 \sigma_2 u = \mathcal{H}_S \psi u - \frac{e\hbar}{2mc} B_k \sigma_k \psi \sigma_3 u$$

We remember that ψ is the matrix defined in (6) and that we have :

(18) $$\sigma_3 u = u \qquad \sigma_1\sigma_2 u = i\sigma_3 u = i u$$

Let us remember the σ_k and $i\sigma_k$ matrices :

(19) $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(20) $$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We note that the $i\sigma_k$ matrices are of course even and that the σ_k matrices are hermitian.

So we can write :

(21) $$\hbar\partial_t \psi (\sigma_1 \sigma_2) u = \mathcal{H}_S \psi u + \frac{e\hbar}{2mc} (B_k i\sigma_k) \psi (i\sigma_3) u$$

where all matrices are now even.

What happens if we realize the equivalent in matrix algebra of a passive rotation of the basis σ, with U real :

(22) $$\sigma'_k = U\sigma_k\tilde{U} \qquad \text{where } U\tilde{U} = \tilde{U}U = 1 \quad \text{(we designate by } \tilde{U} \text{ the hermitian transform)}$$

(22 bis) $$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \tilde{U} = U^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What happens to the different other elements :

– the $\mathcal{H}_S$ scalar operator doesn't change ; neither the ∂_t operator ;

– the $(B_k\, i\sigma_k)$ term is an *intrinsic* even matrix (bivector) ; its components change but globally it doesn't ;

– the spinor matrix ψ becomes $\psi U = \psi'$ (see " Some additional remarks ") ; u becomes $\tilde{U}u = u'$: we verify $\psi u = \psi' u'$.

Thus we obviously can write (21) as :

(23) $$\hbar\partial_t \psi U \left(\tilde{U}\sigma_1 \sigma_2 U\right)\tilde{U}u = \mathcal{H}_S \psi U\tilde{U} u + \frac{e\hbar}{2mc} (B'_k i\sigma'_k) \psi U \left(\tilde{U} i\sigma_3 U\right)\tilde{U}u$$

(24) $$\hbar\partial_t \psi' (\sigma'_1 \sigma'_2) u' = \mathcal{H}_S \psi' u' + \frac{e\hbar}{2mc} (B'_k i\sigma'_k) \psi' (i\sigma'_3) u'$$

We can rewrite (21) and (24) :

(25) $$\left[\!\!\left[\hbar\partial_t \psi (\sigma_1 \sigma_2) - \mathcal{H}_S \psi - \frac{e\hbar}{2mc} (B_k i\sigma_k) \psi (i\sigma_3) \right]\!\!\right] u = 0$$

(26) $$\left[\!\!\left[\hbar\partial_t \psi' (\sigma'_1 \sigma'_2) - \mathcal{H}_S \psi' - \frac{e\hbar}{2mc} (B_k i\sigma_k) \psi' (i\sigma'_3) \right]\!\!\right] u' = 0$$

Now can we simply suppress the primes in the brackets in (26) ? It seems that is precisely what Somaroo tells us when describing his A and B interpretations of rotations (page 14). Look at ψ' : it has the same structure than ψ and is the unknown function ; we can call it ψ without loss of generality. As for the $(i\sigma)$,s it would seem obvious to any seasoned Clifford algebrist that after having done the transformation defined by U one can validly express the σ',s in the new basis by the the initial σ matrices. That is simply the transposition to $SU(2)$ of a well-known propriety of vector rotations. Happily I was reassured in that by a french high level group theory lesson (Bernard Delamotte : " Un soupçon de théorie des groupes " page 39), which one can find on Internet.

Thus we can write :

(27) $$\left[\!\!\left[\hbar\partial_t \psi\, (\sigma_1 \sigma_2) - \mathcal{H}_S\, \psi - \frac{e\hbar}{2mc}\, (B_k\, i\sigma_k)\psi(\,i\sigma_3\,) \right]\!\!\right] u' = 0$$

which with (25) justifies :

(28) $$\hbar\partial_t \psi\, (\sigma_1 \sigma_2) - \mathcal{H}_S\, \psi - \frac{e\hbar}{2mc}\, (B_k\, i\sigma_k)\psi(\,i\sigma_3\,) = 0$$

and the equivalent GA equation.

I am much more happy with that than with ideals and idempotents ...

An unbelievably simple demonstration.

We know that equation (27) must be true, but some elements of the above demonstration might be hard to swallow. Nevertheless I propose an even simpler one. If it is true why has nobody already mentioned it ? If it is false where is the error ?

Equation (21) is an obvious consequence of the P-S equation (1) and of the fact that all matrices in the term between brackets in (27) are even matrices of Mat(2, $\mathcal{C}$). If we take a deeper look at that term, we might conclude that it could formally be written as a matrix of linear, differential or not, operators acting on $(\psi_1,\, \psi_2)$ and their complex conjugates acting on $(\bar{\psi}_1\,, \bar{\psi}_2)$. More precisely one could write (25) as an *even operator matrix*[2] :

(29) $$\Phi\, u = \begin{pmatrix} \Phi_1 & -\bar{\Phi}_2 \\ \Phi_2 & \bar{\Phi}_1 \end{pmatrix} u = 0$$

which implies :

(30) $$\Phi_1 = 0 \qquad \Phi_2 = 0 \qquad \text{and thus} \qquad \bar{\Phi}_1 = 0 \qquad \bar{\Phi}_2 = 0$$

If now we multiply by u' :

(31) $$\Phi\, u' = \begin{pmatrix} \Phi_1 & -\bar{\Phi}_2 \\ \Phi_2 & \bar{\Phi}_1 \end{pmatrix} u' = 0$$

Thus, as developped above, the 4 differential equations are reduced to those expressed by (30), which justifies (25) and (2).

Some additional remarks.

A subject which appears often in Wikipedia talks is that bizarre idea of spinors being square roots of vectors ! It seems linked to that other often badly explained notion, where we are told that vectors and spinors obey different rotation rules : if we rotate a spinor by a 2π angle it is supposed to change sign , so we must rotate it by 4π to restore the initial spinor. Such ideas much publicized in vulgarization articles, shed a mysterious light on spin and spinors.

2. One must note here that Φ_1 is function only of ψ_1 and Φ_2 only of ψ_2 .

As I naïvely learned about spinors first through GA techniques I did not succumb to spiritist visions ! Well I must admit that initially I stumbled at formulas like $\psi\varphi$ supposed to represent the rotation of a spinor φ by a spinor ψ , where one would expect $\psi\varphi\tilde{\psi}$. Indeed if you rotate actively any multivector of any grade with a rotor φ you should write $\varphi A\tilde{\varphi}$. But if, as it happens characteristically with the notion of spin in QM, you have to give an additional rotation to an *intrinsic* element of a physical system, *that has already be defined by a rotation from a reference basis* , you get $\psi\varphi A\tilde{\varphi}\tilde{\psi} = \varphi' A\tilde{\varphi}'$. That equation leads to $\varphi' = \psi\varphi$ and thus to the - in my opinion - false idea that spinors are rotated differently than vectors ! The reality of physics is that the spinors always sandwich characteristic elements ; they are not attached to a specific reference basis but sit between different reference systems, or active positions of the physical system.

But let us take a closer look at that rotation question first in GA . We must note that the relation :

(32) $$U' = VU$$

where U, V, U' are rotors , that is unitary spinors in the GA algebra $\mathcal{G}^3$ constructed on the real space $\mathbb{R}^3$, has nothing to do with quantum mechanics. It represents the action of a rotor on its group, which gives an other rotor. We may call rotation every isometry which is not a translation and which preserves orientation, but the geometric signification is completely lost as shown by the following simple example :

(33) $$[\cos(\theta/2) + In\sin(\theta/2)][\cos(\varphi/2) + In\sin(\varphi/2)] = \cos(\theta/2 + \varphi/2) + In\sin(\theta/2 + \varphi/2)$$

Obviously neither the scalar nor the bivector part on the right could be simply explained by a rotation. And it would look worse of course if we had chosen the general case with different rotation axes.

Instead of that if we write :

(34) $$U' = Vab\tilde{V} = Va.b\tilde{V} + Va\wedge b\tilde{V} = a.b + Va\wedge b\tilde{V} = a.b + iVn\tilde{V}$$

we see, as expected, that the scalar part of (ab) is not changed and that the transformation of the bivector part is expressed by the rotation of its dual vector (**by an angle θ , not** $\theta/2$).

Thus to interprete the relation (32) as a rotation of U by an angle given by V has in my opinion no useful mathematical significance. That is only a question of vocabulary. What remains of course is the fact that if $\theta/2 = 2\pi/2 = \pi$ we get $V = -1$ (!)[3] and $U' = -U$. I will not try to discuss here if that change of sign, which has no incidence on the results of double sided quantum operations, could nevertheless be detected.[4]

Let us now work in standard Clifford algebra Cl_3 . We define first a class $\mathcal{M}$ of (2,2) hermitian traceless matrices of the form :

(35) $$M = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

In the same class $\mathcal{M}$ we find the well known Pauli matrices :

(36) $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus we can write :

(37) $$M = x\sigma_x + y\sigma_y + z\sigma_z$$

3. The fact that usually only that particular value is considered explains perhaps why the non geometrical significance is never mentioned. It goes unnoticed.

4. The practical non quantum mechanical entanglement experiences (plate trick, belt trick, ...) are not easy to explain in mathematical terms. I doubt that they presuppose SU(2) complex matrices ...

Obviously there exists a bijective relation between the $\mathcal{M}$ matrices and the vectors in the real space $\mathbb{R}^3$. The σ matrices can be identified with the unit orthogonal basis vectors in $\mathbb{R}^3$.

We define also the SU(2) group which consists of all matrices of the following form :

$$(38) \qquad U=\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad \text{with} \qquad U^{-1}=\bar{U}^T=\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \qquad \alpha\bar{\alpha}+\beta\bar{\beta}=1$$

The unit matrix 1 and the $i\sigma$ matrices are elements of SU(2), where they form a basis.

In $\mathbb{R}^3$ the rotation operators are the well known rotation (3,3) matrices which are elements of the so-called SO(3) group. They act on the vectors in $\mathbb{R}^3$.

It is easy to show that the SU(2) matrices define a 2 dimensional representation of rotations, up to a sign[5] :

$$(39) \qquad M'=UM\tilde{U}=(-U)M(-\tilde{U})$$

where the general rotation matrix is given by :

$$(40) \qquad U=\begin{pmatrix} \cos(\theta/2)+i\,n_z\sin(\theta/2) & (in_x+n_y)\sin(\theta/2) \\ (in_x-n_y)\sin(\theta/2) & \cos(\theta/2)-i\,n_z\sin(\theta/2) \end{pmatrix}$$

that is :

$$(41) \qquad U=\cos(\theta/2)\,\mathbf{1}+i(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)\sin(\theta/2) \qquad \text{with} \quad |n|=1$$

We notice that unlike the matrices in $\mathbb{R}^3$, here the operator acts double-sidedly. That explains the $\theta/2$ instead of θ ! Where is the mystery ?

The SU(2) matrices operate on 2 dimensional complex column vectors of a spin space , that is :

$$(42) \qquad \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}=U\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \text{let us say} \qquad \Psi'=U\Psi$$

It is interesting to note that, as the U matrices are even, in the sense defined by GA, (42) implies also :

$$(43) \qquad \begin{pmatrix} z_1' & -\bar{z}_2' \\ z_2' & \bar{z}_1' \end{pmatrix}=U\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \qquad \text{that is} \qquad \psi'=U\psi$$

Here ψ can of course be defined by a formula like (41), or directly by a matrix like (5).

Let us take a look at a very simple example. In spin space two obvious orthonormal basis vectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ can be considered as the first column of a very simple $\mathcal{M}$ matrix , that is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which according to the definition (35) corresponds to the σ_3 basis vector in $\mathbb{R}^3$. In SU(2) it corresponds to the σ_3 matrix .

If we rotate that vector in $\mathbb{R}^3$ by an angle (θ) around the σ_2 basis vector we get[6] :

$$(44) \qquad [\cos(\theta/2)1+\sin(\theta/2)i\sigma_2]\sigma_3[\cos(\theta/2)1-\sin(\theta/2)i\sigma_2]=$$
$$[\cos(\theta)1+\sin(\theta)i\sigma_2]\sigma_3=\cos(\theta)\sigma_3-\sin(\theta)\sigma_1$$

which is a classical rotation in $\mathbb{R}^3$.

But if we rotate the same vector one - sidedly by ($\theta/2$) in the spin space, we get :

5. Thus SU(2) is said to be a double cover of SO(3).

6. Of course we do the practical calculation with GA !!

(45) $$[\cos(\theta/2)1+\sin(\theta/2)i\sigma_2]\sigma_3=\cos(\theta/2)\sigma_3-\sin(\theta/2)\sigma_1$$

which as the rotation of a vector Ψ (first column of the matrix ψ) has to be written :

(46) $$[\cos(\theta/2)1+\sin(\theta/2)i\sigma_2]\begin{pmatrix}1\\0\end{pmatrix}=\cos(\theta/2)\begin{pmatrix}1\\0\end{pmatrix}-\sin(\theta/2)\begin{pmatrix}0\\1\end{pmatrix}$$

That looks indeed like a rotation, but what happens now if we do the same thing about the σ_1 basis vector. We get :

(47) $$[\cos(\theta/2)1+\sin(\theta/2)i\sigma_1]\sigma_3[\cos(\theta/2)1-\sin(\theta/2)i\sigma_1]=$$
$$[\cos(\theta)1+\sin(\theta)i\sigma_1]\sigma_3=\cos(\theta)\sigma_3+\sin(\theta)\sigma_2$$

and :

(48) $$[\cos(\theta/2)1+\sin(\theta/2)i\sigma_1]\begin{pmatrix}1\\0\end{pmatrix}=\cos(\theta/2)\begin{pmatrix}1\\0\end{pmatrix}+i\sin(\theta/2)\begin{pmatrix}0\\1\end{pmatrix}$$

Obviously the geometrical meaning has again been lost ! In classical mechanics making use of the SU(2) matrices would only generate unnecessary complications.[7]

Why then all the fuss about that change of sign when $\theta = 2\pi$? To try to find an answer we must now consider quantum physics.

The answer is stunningly simple. In standard quantum physics we do not (or should not ? , see " Spineurs .... encore ") use the direct identification between for example the σ_3 spinor (or Σ_3 spin vector, which is the first column of the σ_3 matrix) and the σ_z unit vector in $\mathbb{R}^3$. What we consider is the *expectation value* of the spin in for example the k-direction, that is :

(49) $$\tilde{\Psi}\,\sigma_k\,\Psi \quad \text{where } \Psi \text{ represents the state vector .}$$

We use the σ_3 matrix in the spin space to define the spin vector. Then of course we get for (spin up) $\Psi_1=\begin{pmatrix}1\\0\end{pmatrix}$:

(50) $$\begin{pmatrix}1 & 0\end{pmatrix}\sigma_3\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=1$$

and for $\Psi_1'=-\begin{pmatrix}1\\0\end{pmatrix}$, where Ψ_1' is the vector in spin space which is supposed to be obtained from Ψ by a $\theta/2=\pi$ rotation :

(51) $$\begin{pmatrix}-1 & 0\end{pmatrix}\sigma_3\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}-1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}=1$$

The expectation value is the same after the above defined *rotation*.

The same result is true for the spin down vector $\Psi_2=\begin{pmatrix}0\\1\end{pmatrix}$.

One must be very careful when expressing the conclusion. In quantum mechanics a rotation of a spinor operator by an angle $\theta/2 = \pi$ changes the sign of that spinor, *in spin space*, but does not change the *expectation value* of the quantum variable on which it acts.

That is where geometric algebra brings important novelties where a true spin vector, or - better - bivector, exists in $\mathbb{R}^3$, and there is no more place for an abstract spin space and SU(2) group. In GA the equations would be written :

(52) $$(1)\,\sigma_3\,(1)=\sigma_3 \qquad \text{that is spin up in } \mathbb{R}^3$$

(53) $$(-i\sigma_2)\,\sigma_3\,(i\sigma_2)=-\sigma_3 \qquad \text{that is spin down in } \mathbb{R}^3$$

But then we enter in the forbidden domain of quantum physics interpretation ... !

G.Ringeisen

June 2013

7. But with GA new interesting possibilities appear even in classical physics.