Foucault's pendulum with Geometric Algebra.

I. Introduction

Foucault's pendulum is a well known but rather difficult subject. A lot of professional and amateur physicists have tried during more than 150 years to contribute to the solution or at least to the explanation of the mouvement of that star linked pendulum ! More than often false ideas were enunciated. The most tempting was indeed to consider that the pendulum must oscillate in a plane which remains fixed in some inertial coordinate system related to the stars, but by comparing two adjacent planes it is easy to verify that such a situation cannot be true except at the Earth poles.

The valuable articles written on the subject can roughly be classified in two categories : first those which establish the mathematical equations of the pendulum's mouvement, simplify them and draw analytically the conclusion that the trace on an earthbound reference coordinate system rotates in the opposite direction of Earth rotation by an angle $\omega \Delta t \sin\lambda$ [1] , second those which try to come to that conclusion with topological arguments without solving or even establishing analytical relations.

In the second, minority category, we find the mathematician and physicist Liouville who, immediately when Foucault presented his experimental results to the French Academy of Science in 1851, argued that these results could be explained by the fact that the pendulum's orientation was sensitive only to the vertical component of the rotational earth velocity at the experimental site. He was right but we do not know the details of his argument. The great Poinsot came also, after revising his initial negative opinion - in 1835 he ruled out any effect of the Coriolis force on a swinging pendulum -, to the conclusion that the pendulum's mouvements could be explained by the sole cinematic facts.

Having read some of those texts I was not completely satisfied, neither with the former where integration of the pendulum's movements necessitate inelegant approximations, nor with the latter where the topological ideas seduced me, but where I felt a lack of rigor in the quick sweeping away of possible dynamic perturbations.

As simultaneously I was always in search of possible applications with Geometric Algebra techniques, I wondered if I could rigorously demonstrate, in some well chosen moving coordinate system, that indeed there subsist no effects due to centrifugal forces or to Coriolis forces, and thus reduce the question of the movement of the pendulum's plane to the topological rotation of a North - South or any other fixed vector in an earthbound reference system.

We will use following systems of coordinates in the Northern Hemisphere :

-- (S1) is a quasi-inertial system whose center coincides with the Earthcenter, orthonormal, right handed, whose axis 3 (unit vector w) is directed towards the Northpole, and whose other axes remain fixed relatively to the stars ; that means that we neglect principally the fact that the Earth rotates around the Sun ;

-- (S2) is an Earthbound orthonormal coordinate system centered at the same origin, whose axis 3 is oriented towards the point where the pendulum is suspended, which we name x (x is a geometric algebra vector , unit vector $\hat{x}$); axis 1 with unit vector u is southward oriented in the plane defined by x and the Earth-axis ; thus axis 2 with unit vector v is always perpendicular to that plane and Eastward oriented ;

-- (S'2) is the (S2) coordinate system translated to the surface of the Earth to materialize in the marble table on which the pendulum plane's rotational move can be observed ; we will not need it for our calculations, but it is important to note that the trace of the pendulum plane observed on that table is parallel to the trace of the same plane which we would observe, if possible, on the (u, v) plane in (S2) ;

1. λ is the angle of latitude, ω the scalar value of the rotational velocity of the Earth, Δt the examined time span.

-- (S3) is the auxiliary coordinate system, with Earth center origin, which we introduce to show with mathematical rigor that one can completely eliminate all forces which could deviate the pendulum's plane from the fixed position it has to observe in (S3) ; that result is obtained by defining (S3) as rotating clockwise, that is in the opposite direction of the Earth rotation, around $x(t)$ with the angular velocity $\omega \sin\lambda$, which is the vertical component of ωw (Earth angular velocity vector) on the observation site at latitude λ .

-- (S'3) is the same system translated to the Earth surface where it rotates clockwise on the surface of the marble table which characterizes (S'2).

We add that the bob of the pendulum is designated by the vector y . Thus the plane of the pendulum is defined by the origin O center of Earth, and the vectors x and y . We have of course always $|y - x| = l$.

II. Some elementary topology.

One of the difficulties of the Foucault pendulum question is the fact that the observer situated in (S'2) has, unless he performs simultaneously some astronomical observations, no valid reference vector - that means fixed in an inertial system - at his disposal. If he engraves on the marble table the projection of the u vector that we have defined at Earth center O, he gets a vector always oriented towards the South pole and thus always perpendicular to the λ latitude cercle. If he had some knowledge of topology he could immediately tell us that his reference vector is **not** parallel transported along the latitude circle, because such a circle is not a geodesic of the sphere[2]. But with geometric algebra we can dispense with such sophisticated notions.

Let us examine what becomes with the basis vectors of (S1) if we rotate them first with the rotation of the Earth, second simultaneously with the clockwise rotation between (S2) and (S3). We are interested only in infinitesimal rotations during the timespan δt .

If we call ξ any vector rigidly fixed in (S1) we get :

(1) $\delta_1 \xi = \xi . (w I) \omega \delta t$ where $w = \hat{x} \sin\lambda - u \cos\lambda$ $I = u v \hat{x}$

(2) $\delta_2 \xi = \xi . (w I - \sin\lambda\, \hat{x} I) \omega \delta t = - \xi . (u I) \cos\lambda\, \omega \delta t$

If we apply these formulas to u and $\hat{x}$ we get :

(3) $\delta_1 u = u . (w I) \omega \delta t = u \wedge w I \omega \delta t = u \wedge \hat{x} I \sin\lambda\, \omega \delta t = v \sin\lambda\, \omega \delta t$

(4) $\delta_2 u = u . (u I) \cos\lambda\, \omega \delta t = 0$

(5) $\delta_1 \hat{x} = \hat{x} . (w I) \omega \delta t = \hat{x} \wedge w I \omega \delta t = - \hat{x} \wedge u I \cos\lambda \omega \delta t = v \cos\lambda \omega \delta t = \delta_2 \hat{x}$

What is important here are the values of the $\delta u's$, and particularly the stationarity of the vectors $(u + \delta_2 u)$. Of course to integrate these elements we must project them on (S'2) and (S'3), because the vector x moves. Then it is easy to see that $\delta_1 u$ can be integrated in the moving (S'2) and $\delta_2 u$ in (S'3). The conclusion is :

-- the vector u projected on (S'2) moves, relatively to the South direction, clockwise with the angular velocity $\omega \sin\lambda$ (equation (3)) [3] ;

-- the same vector projected on (S'3) stays fixed relatively to any reference vector engraved in the rotating disk (equation (4)) ;

2. Only along geodesics - that is straight lines on a riemanian surface - remains the angle constant between the parallel transported vector and the trajectory.

3. The double interpretation which is expressed by $u.(w I) = u.(\hat{x} I)\sin\lambda$ may at first sight surprise some readers. It shows how easy it is to rediscover with GA forgotten geometrical facts.

-- of course by construction these results are coherent.

We may add an interesting observation : what happens when we project the same vector $u(t_0)$ on the successive (S'2) at the points $x(t_0)$ and $x(t_0+\delta t)=x(t_0)+\delta x$? At both locations we find that the projection is equal to $u(t_0)$ within $\omega^2\delta t^2$ precision. It is easy to check with GA ; the first projection is obvious, the second is given by :

(6) $$u\wedge(x+\delta x)(x+\delta x)=u\wedge x(x+\delta x)+u\wedge\delta x(x+\delta x)$$
$$=(u\wedge x).x+u\wedge x\wedge\delta x+u\wedge\delta x\wedge x+(u\wedge\delta x).(x+\delta x)$$
$$=u+0+0+u(\delta x)^2=u$$

But we know that the engraved South direction has rotated in (S1) counterclockwise by an angle $\sin\lambda w\delta t$; that means that the projected $u(t_0)$ vector now makes an angle $(\pi/2-\sin\lambda\omega\delta t)$ with the latitude circle. We know also that the geometric construction of a parallel transport on the sphere implies the projection on its tangent planes of a vector parallel transported in the embedding Euclidean space. Thus we may conclude that we just executed an infinitesimal parallel transport of the vector $u(t_0)$ on the sphere.

III. Elements of the dynamics of the pendulum.

Now we guess that the angle between the projected - on (S'2) and (S'3) - vector $u(t)$ and the trace of the pendulum's swinging plane on the successive tangent planes will remain constant. That shall be true of course in any coordinate system, but it should be particularly easy to establish in a system where both the centrifugal forces and the Coriolis forces acting on the bob would be reduced to almost nothing, if not zero. Thus we will study the different real and so-called fictitious forces which appear in the system (S3). The trace in (S'3) is the projection of the trace in (S3).

We can exclude from our study both the gravitation force which is oriented to the center O of the Earth, and the tension of the wire which originates in x . None of them can influence the direction of the bob's trajectory.[4]

The centrifugal force is expressed by :

(7) $$f_{C1}=-\omega^2\cos^2\lambda[y.(uI)].(uI)=-\omega^2\cos^2\lambda[y.(v\wedge\hat{x})].(v\wedge\hat{x})$$
$$=\omega^2\cos^2\lambda(y.vv+y.\hat{x}\hat{x})=\omega^2\cos^2\lambda(y-x).vv+\omega^2\cos^2\lambda|y|\hat{y}.\hat{x}\hat{x}$$

Only the term $\omega^2\cos^2\lambda(y-x).vv$ has to be examined.

The Coriolis force is expressed by :

(8) $$f_{C2}=-2\omega\cos\lambda v_r.(-uI)=2\omega\cos\lambda[\dot{y}+\omega\cos\lambda y.(uI)].(uI)$$

where we name v_r the relative velocity vector of the pendulum in (S3) and (S'3). We can substitude to v_r the relative velocity vector v_p of the pendulum in (S'2) by writing :

(9) $$\dot{y}-\dot{x}=v_p+\omega(y-x).(wI)$$

Thus we get, by using equations (1) repeatedly and (9) :

(10) $$f_{C2}=2\omega\cos\lambda[(v_p+\dot{x}+\omega(y-x).(wI)+\omega\cos\lambda y.(uI)].(uI)$$
$$=2\omega\cos\lambda[v_p+\omega\sin\lambda\, y.(\hat{x}I)].(uI)$$

4. We suppose of course that the bob's mouvement is adequately initialized, with velocity nil. We have intentionnaly renounced to include the centrifugal force in the "gravitational force".

$$= 2\omega\cos\lambda[v_p + \omega\sin\lambda(y-x).(\hat{x}I) + \omega\sin\lambda\, x.(\hat{x}I)].(uI)$$

$$= 2\omega\cos\lambda[v_p + \omega\sin\lambda(y-x).(\hat{x}I)].(uI)$$

$$= 2\omega\cos\lambda v_p.(uI) + \omega^2\sin 2\lambda[(y-x).(\hat{x}I)](uI)$$

$$= -2\omega\cos\lambda\, v_p\,.\,\hat{x}\,v + 2\omega\cos\lambda v_p\,.v\,\hat{x} + \omega^2\sin 2\lambda(y-x).u\,\hat{x}$$

Here only the term $-2\omega\cos\lambda v_p\,.\,\hat{x}\,v$ has to be examined.

We must insist on the fact that until this point we have made no approximate evaluation. The relations (7), (8), (10) are strictly in accordance with the theory, without any simplification. Of course we have admitted that the Earth is a perfect sphere, and have neglected friction forces which can be adequately compensated in the experimental devices. We have also neglected the fact that the (S1) coordinate system cannot be an absolutely perfect inertial system.

IV. Conclusion.

Let us now elaborate some numerical values :

radius of Earth $= 6.4\ 10^8$ cm

gravity $= 981$ cm s^{-2}

$|x-y| = 300$ cm

half-swing width of pendulum $= 100$ cm

highest velocity of the bob $= 183$ cm s^{-1}

lowest velocity of the bob $= 0$

intermediate velocity of the bob $\simeq 130$ cm s^{-1}

$v_p\,.\,\hat{x}$ at intermediate position $\simeq 130 \times 0.24 = 31.2$ cm s^{-1}

at 45 degrees of latitude we have $\cos\lambda = \sin\lambda = 0.71$ $\cos^2\lambda = 0.5$

$\omega = 7.27\,10^{-5}$ rad s^{-1} $\omega^2 = 5.28\ 10^{-9}$ rad s^{-2}

So we get for the centrifugal term $\omega^2\cos^2\lambda(y-x).vv$ a value which oscillates between 0.0 and a possible maximal value of $5.28 \times 0.5\,10^{-9}$, when the pendulum oscillates parallel to v , which is of course completely negligible.

For the Coriolis term $2\omega\cos\lambda v_p\,.\,\hat{x}\,v$ the comparison is to be made with the tangential acceleration of the bob which, as can be easily verified on the simple pendulum equations, is a vector with norm $g\sin\theta$ parallel to v_p in the pendulum oscillation plane. Thus we get for an intermediate position approximatively[5] the scalar ratio

$2\omega\cos\lambda v_p.\hat{x}\ /(g\sin\theta\cos\theta) = 2 \times 7.27 \times 0.71 \times 31.2 \times 10^{-5}/(981 \times 0.24 \times 0.971) = 5.9\,10^{-5}$ which is also negligible, but justifies the fact that only the centrifugal term is usually incorporated in gravity.

Thus we can safely draw the conclusion that, appreciated in (S3) and thus in (S'3), the angle between the trace of the pendulum and the projected $u(t)$ vector remains constant. And as we have demonstrated that the same $u(t)$ vector rotates clockwise (seen from above) with the angular velocity $\omega\sin\lambda$, we draw the same conclusion for the trace. And the nice fact, which explains the curiosity success of all existing museum Foucault pendulums, is that we can see the plane rotating whereas we cannot observe the $u(t)$ vectors ! The engraved South direction enduces us in error ...

5. I calculated the elements for a maximal height of the bob of 17.2 cm and an intermediate height of 8.6 cm , which is coherent with the 100 - 300 configuration .

For the mathematical oriented reader we add that then we have also proved that the trace of the pendulum on the Earth surface represents the parallel displacement of a vector along a latitude circle[6] We hope that he appreciates the efficient contribution of GA techniques to that task. Note that we have defined five different coordinate systems, but never used them as such it should or could be ! Indeed without GA I would never have tried such an approach.

G.Ringeisen

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Addendum.

At the end of section II we have proved in detail that if at each instant t we project the parallel transported (in R^3) vector $u(t)$ on two infinitesimaly near tangent planes along a latitude circle of the sphere we get the same vector $u(t)$ within $\omega^2\,\delta t^2$ precision. We can do more ! We will prove that if we name $z(t)$ the trace direction vector in the plane defined by $(u,\,v)$, and if we project that trace vector on the same tangent planes as above, then the angle between the projected vectors $u(t)$, $z(t)$ will remain constant within the precision $\omega^2\delta t^2$.

We take again the tangent planes at x and $x+\delta x$. We define $z(t)$ by :

(11) $$z=\cos\alpha\, u+\sin\alpha\, v$$

The projection of z on the tangent plane at x is obviously z , which we check by :

(12) $$z\wedge\hat{x}\,\hat{x}=(z\wedge\hat{x}).\hat{x}=z$$

The projection on the tangent plane at $x+\delta x$ is :

(13) $$\begin{aligned} z\wedge(\hat{x}+\delta\hat{x})(\hat{x}+\delta\hat{x}) &= z\wedge\hat{x}\,(\hat{x}+\delta\hat{x})+z\wedge\delta\hat{x}\,(\hat{x}+\delta\hat{x}) \\ &= z\wedge\hat{x}\hat{x}+z\wedge\hat{x}\,\delta\hat{x}+z\wedge\delta\hat{x}\,\hat{x}+z\wedge\delta\hat{x}\,\delta\hat{x} \\ &= z+z\wedge\hat{x}\,\wedge\delta\hat{x}+z\wedge\delta\hat{x}\,\wedge\hat{x}+(z\wedge\hat{x}\,).\delta\hat{x}+(z\wedge\delta\hat{x}\,).\,\hat{x}+(z\wedge\delta\hat{x}).\,\delta\hat{x} \\ &= z+0-z\,.\delta\hat{x}\;\hat{x}-0+o(\omega^2\delta t^2)=z-(\cos\alpha\,u+\sin\alpha\,v).v\,\hat{x}\cos\lambda\,\omega\delta t+o(\omega^2\delta t^2) \\ &= z-\omega\,\delta t\sin\alpha\,\cos\lambda\,\hat{x}+o(\omega^2\delta t^2) \end{aligned}$$

Now to evaluate the angle we get :

(14) $$[z-\omega\,\delta t\sin\alpha\,\cos\lambda\,\hat{x}+o(\omega^2\delta t^2)].[u+u\,o(\omega^2\delta t^2)]=z\,.u+o(\omega^2\delta t^2)$$

which demonstrates our proposition[7].

Additionally I submit the following idea : we can admit that in the δt time interval the movement of the bob can be studied in the inertial (S1) coordinate system ; as no perturbing forces are at work in that situation[8], the pendulum's plane doesn't rotate in (S1) during that infinitesimal timespan and thus the trace on the (S2) system moves relatively to the South mark, clockwise, with the same velocity as the projected $u(t)$ vector. That simple constant rotation velocity can of course be integrated in (S'2).

Thus we have proved the behaviour of the Foucault pendulum without any approximate numerical calculus and without integrating it analytically.

6. .Because the trace will rotate along the latitude cercle with the same angular velocity than the projected u(t) vector.

7. Who will try to do that in standard Gibbs vector algebra ?

8. More rigourosly that means within an $(\omega^2\delta t^2)$ precision.

For other, classical but sophisticated, articles based on topological arguments see for instance on Internet :

-- Foucault pendulum through basic geometry. Jens/HsingChi von Bergmann

-- Rotation of the swing plane of Foucault's pendulum and Thomas spin precession : Two faces of one coin. M. I. Krivoruchenko

-- Geometry and the Foucault pendulum. John Oprea