Time dilatation easy with Geometric Algebra.

In a preceding article we tried to give a new interpretation of time dilation in Special Relativity. One of our conclusions was the physicists work mostly with coordinates , tensorial or not, and neglect vectorial methods. That impended many students and even professors to acquire a deeper knowlege of the Minkowski space. But now that the Geometric Algebra has been revivified we have a very efficient mathematical method at our disposal.

Let us see how that works.

(1) $\quad e_0^2 = {e'_0}^2 = 1 \qquad e_1^2 = {e'_1}^2 = -1 \qquad e_0 \,.\, e_1 = e'_0 \,.\, e'_1 = 0$

These relations between the unit vectors define entirely the Minkowski space , here reduced to pseudo-euclidian plane. It is important to note that whatever the choice of the reference frame the isotropic directions are unchanged (null vectors).

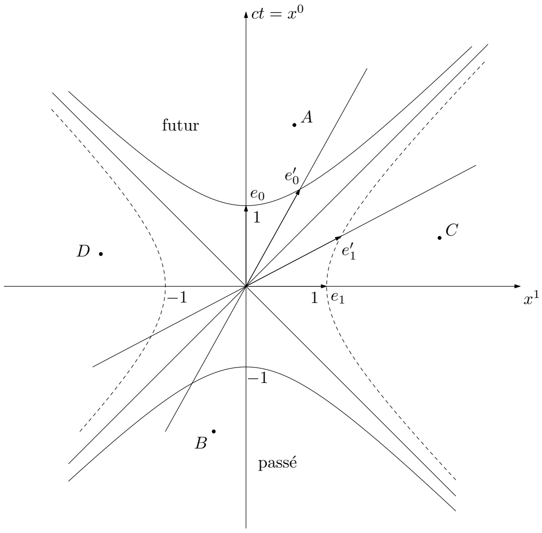

Figure 1.

Now have to etablish the Lorentz tranformation between two frames. That will be very easy with the most fundamental relation in GA , between the geometric product of two vectors and its components (symetric and antisymetric) :

(2) $\quad e'_0 \, e_0 = e'_0 \,.\, e_0 + e'_0 \wedge e_0 \quad \longrightarrow \quad e'_0 = e'_0 \,.\, e_0 \, e_0 + e'_0 \wedge e_0 \, e_0$

(3) $\quad e'_0 \,.\, e_0 = \Gamma$

We define Γ by (3) and we guess that Γ is indeed the Lorentz factor (demonstration follows).

We notice that the vector e'_0 is decomposed into a vector parallel to e_0 , and a vector orthogonal to it :

(4) $\quad e'_0 \wedge e_0 \, e_0 = e'_0 \, e_0 \, e_0 - e'_0 \,.\, e_0 \, e_0 = e'_0 - \Gamma e_0 \quad \longrightarrow \quad (e'_0 \wedge e_0 \, e_0) . e_0 = 0$

We guess now that $\bar{v}$ is the euclidian velocity wich represents the movement of the primed system (demonstration follows) :

(5) $$\bar{v} = \frac{e_0' \wedge e_0}{e_0' \,.\, e_0}$$

(6) $$e_0'\, e_0 = \Gamma\left(1 + \frac{e_0' \wedge e_0}{e_0' \,.\, e_0}\right) = \Gamma(1+\bar{v})$$

(7) $$1 = e_0'\, e_0\, e_0\, e_0' = \Gamma^2(1+\bar{v})(1-\bar{v}) = \Gamma^2(1-\bar{v}^2)$$

(8) $$\Gamma = (1-\bar{v}^{\,2})^{-1/2}$$

Thus , as we guessed , Γ is in accordance with the definition of $\bar{v}$.

Take now a look at figure 2 :

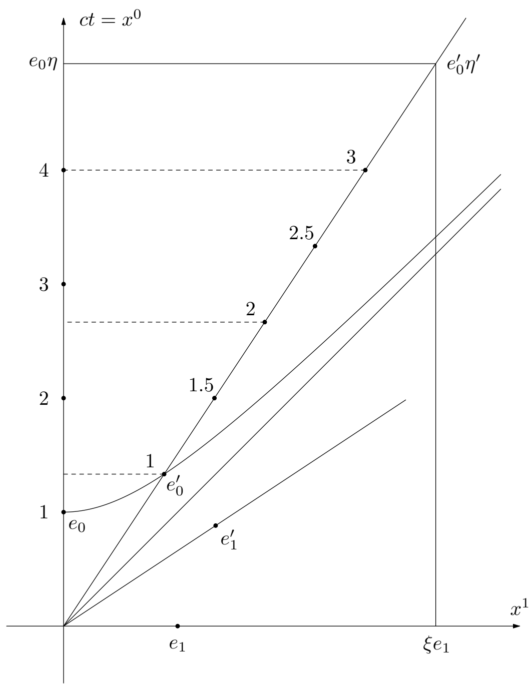

Figure 2.

Following calculations are obvious :

(9) $$x = e_0\,\eta + e_1\,\xi = e_0'\,\eta' + 0$$

(10) $$x.e_0 = \eta = \Gamma\,\eta' \qquad \longrightarrow \qquad \eta' = \Gamma^{-1}\eta$$

(11) $$x \wedge e_0 = e_1 \wedge e_0\,\xi = e_0' \wedge e_0\,\eta' = e_0' \wedge e_0\,\Gamma^{-1}\eta$$

(12) $$\xi = \eta\,\frac{e_0' \wedge e_0}{e_0' \,.\, e_0}\,\frac{1}{e_1 e_0} = \eta\,\bar{v}\,\frac{1}{e_1 e_0}$$

(13) $$|\bar{v}\,| = \bar{v}\frac{1}{e_1 e_0}$$

The factor e_1e_0 , whose norm is 1 , is necessary because $\bar{v}$ is a bivector in GA , and a relative vector in the space-time split (in fact here a scalar).

Observe we have not told a single word of simultaneity. We think it is a mathematicle notion , not a physical one in space-time. We spend too much time trying to resolve false paradoxes ...

To resume , all the very basic results of special relativity , time dilation , length contraction , time and length units , are contained in a single GA equation :

$$e'_0\, e_0 = e'_0 \,.\, e_0 + e'_0 \wedge e_0$$

That justifies a new lecture of many pages of specialised scientific literature ...

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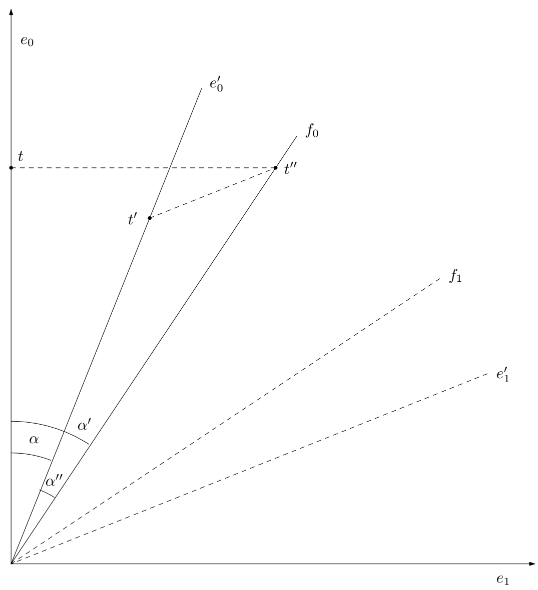
Geometric Algebra for Physicists (Doran , Lasenby)

Space-time Physics (Hestenes)

Time dilation in Special Relativity (Ringeisen)

Addendum. (see phymatheco.pagesperso-orange.fr/Timedila.pdf)



$t'' = \gamma'^{-1} t = \gamma''^{-1} t'$

Figure 3.

Geometric Algebra (GA) should be the preferred mathematical instrument to study relativity. Its vectorial approach is ideal to manipulate the hyperbolic geometry which constitudes the core of the spacetime algebra.

Start again with figures 1 and 2. Call α the parameters. We get :

(14) $e'_0 = \gamma e_0 + \gamma\beta e_1 = \exp^{\alpha e_1 e_0} e_0$ $\qquad$ $e'_1 = \gamma\beta e_0 + \gamma e_1 = \exp^{\alpha e_1 e_0} e_1$

and more generally :

(15) $e'_\mu = R e_\mu \tilde{R}$ $\qquad$ $R = \exp^{\alpha e_1 e_0/2}$

with :

(16) $\exp^{\alpha e_1 e_0} = \mathrm{ch}\alpha + e_1 e_0 \,\mathrm{sh}\alpha$

That means we can introduce an « angle » α with :

(17) $\beta = v = \mathrm{th}\alpha$ $\qquad$ $\gamma = \mathrm{ch}\alpha = (1 - v^2)^{-1/2}$ $\qquad$ $\beta\gamma = \mathrm{sh}\alpha$

We recall :

(18) $e_0 . e'_0 = e_0 . [(\gamma + \gamma\beta e_1 e_0) e_0] = \gamma$

Thus (figure 2) :

(19) $e_0 . (\eta' e'_0) = \gamma\eta'$ $\qquad$ $\eta' = \gamma^{-1}\eta$

We define (figure 3) the « angle » α' :

(20) $f_0 = \exp^{\alpha' e_1 e_0} e_0$ $\quad$ with $\quad$ $\mathrm{ch}(\alpha') = \gamma'$ $\quad$ $\mathrm{sh}(\alpha') = \beta'\gamma'$ $\quad$ $\beta' = w$

As we see , in the falsely euclidian figure , e_0 is transformed in e'_0 by the hyperbolic « rotation » α , and in f_0 by the « rotation » α' . That justifies our graphisme. Of course in numerical traduction α and α' are not angles !

Now we can find an elegant way to describe a change of reference frames. We introduce (figure 3) the new frame (e'_0, e'_0) with the « angle »α , and then call α'' the « angle » between e'_0 and f_0 . We call u the velocity f_0 relative to (e'_0, e'_1) . That is :

(21) $f_0 = \exp^{\alpha'' e'_1 e'_0} e'_0$ $\quad$ with $\quad$ $\mathrm{ch}(\alpha'') = \gamma''$ $\quad$ $\mathrm{sh}(\alpha'') = \beta''\gamma''$ $\quad$ $\beta'' = u$

By (14) we have :

(22) $e_0 = \exp^{-\alpha e_1 e_0} e'_0$ $\qquad$ $e_1 = \exp^{-\alpha e_1 e_0} e'_1$

Thus by (20) :

(23) $f_0 = \exp^{(\alpha' - \alpha) e_1 e_0} e'_0$

We know :

(24) $e_1 e_0 = e'_1 e'_0$ $\quad$!!

Thus :

(25) $f_0 = \exp^{(\alpha' - \alpha) e'_1 e'_0} e'_0$ $\qquad$ $\mathrm{ch}(\alpha' - \alpha) = \gamma''$ $\quad$ $\mathrm{sh}(\alpha' - \alpha) = \gamma''\beta''$ $\quad$ $\beta'' = u$

and :

(26) $\alpha'' = \alpha' - \alpha$ $\qquad$!!

The α parameters are additive.