Two theorems on rotations in R^3.

I would like to show how efficient GA is in representing rotations in R^3 , and how easy it is with that tool to demonstrate some not so obvious theorems.

The first one is well-known but usually admitted without citing a proof, the second one was probably known in the 18th, 19th centuries but seems to have been forgotten by modern mathematicians working with group theory and rotation generators. I found nowhere any mention of it. With those sophisticated abstractions it is easy to forget the very geometrical meaning of rotations.

Theorem 1 : *Two successive finite rotations in R^3 are equal to one unique rotation if and only if the two axes, and of course the third also, are concurrent.*

There must certainly exist more than one pure geometric demonstration of that fact but it is very instructive to achieve it with GA.

Let us characterize a rotation $\mathfrak{R}_i$ by an anchor point a_i , an angle θ_i and the direction of the axis u_i , where $u_i^2=1$. So the associated rotor is :

(1) $$R_i = \cos\frac{\theta_i}{2} + I u_i \sin\frac{\theta_i}{2}$$

Let us transform any arbitrary chosen point x_0 first by the rotation $\mathfrak{R}_1$ followed by $\mathfrak{R}_2$. We expect that composite operation to be equal to a rotation $\mathfrak{R}_3$.

Thus we should have :

$$x_1 = \tilde{R}_1 (x_0 - a_1) R_1 + a_1$$

(2) $$x_2 = \tilde{R}_2\{\tilde{R}_1(x - a_1)R_1 + a_1 - a_2\}R_2 + a_2$$

$$x_2 = \tilde{R}_3(x_0 - a_3)R_3 + a_3$$

That is :

(3) $$\tilde{R}_2\{\tilde{R}_1(x_0 - a_1)R_1 + a_1 - a_2\}R_2 + a_2 = \tilde{R}_3(x_0 - a_3)R_3 + a_3$$

That unique relation contains everything we need to know :

– a feasibility condition,

– the formula of the R_3 rotor, that is u_3 and θ_3 ,

– the anchor point a_3 .

It must be noted here that of course the points a_1, a_2, a_3 can be freely moved along their respective axes, as :

$$(a + \lambda u) - \tilde{R}(a + \lambda u)R = (a - \tilde{R}\, a R)$$

First we observe that as we can choose any initial point x_0 , we must have :

(4) $$R_3 = R_1 R_2 \qquad \tilde{R}_3 = \tilde{R}_2 \tilde{R}_1$$

That gives :

(5) $$\cos\frac{\varphi_3}{2} = \cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} - u_1 \,.\, u_2 \sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}$$

(6) $$I u_3 \sin\frac{\varphi_3}{2} = I u_1 \sin\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} + I u_2 \sin\frac{\varphi_2}{2}\cos\frac{\varphi_1}{2} - u_1 \wedge u_2 \sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}$$

Now (3) transforms to :

(7) $$(a_3 - a_1) - \tilde{R}_3(a_3 - a_1)R_3 = (a_2 - a_1) - \tilde{R}_2(a_2 - a_1)R_2$$

$$R_3(a_3 - a_1) - (a_3 - a_1)R_3 = R_1\{R_2(a_2 - a_1) - (a_2 - a_1)R_2\}$$

$$Iu_3(a_3 - a_1) - (a_3 - a_1)Iu_3 = R_1\{Iu_2(a_2 - a_1) - (a_2 - a_1)Iu_2\}$$

(8) $$u_3 \wedge (a_3 - a_1) = R_1\, u_2 \wedge (a_2 - a_1)$$

At the left we have a bivector ; thus the scalar part at the right must be equal to zero :

$$< u_3 \wedge (a_3 - a_1) > = 0 = < R_1\, u_2 \wedge (a_2 - a_1) > = (Iu_1).\{u_2 \wedge (a_2 - a_1)\}$$

(9) $$0 = u_1 \wedge u_2 \wedge (a_2 - a_1)$$

That means that $(a_2 - a_1), u_2, u_1$ are coplanar. Thus the two rotation axes must be concurrent ; we can move the anchor points to the common point which we take as the origin ; then we get also $a_3 = 0$. The three axes are concurrent ; equations (5) and (6) give the complete solution of the problem. If we need to calculate a_3 we get from (8) :

$$u_3 \wedge (a_3 - a_1) = u_3 \wedge (a_3 - a_1)_\perp = u_3\,(a_3 - a_1)_\perp = R_1\, u_2 \wedge (a_2 - a_1)$$

(10) $$(a_3 - a_1)_\perp = u_3\, R_1\, u_2 \wedge (a_2 - a_1)$$

(11) $$u_3 \wedge (a_3 - a_1)_\parallel = 0$$

(12) $$a_3 - a_1 = u_3\,\{R_1\, u_2 \wedge (a_2 - a_1) + \lambda\}$$

Theorem 2 : *Three or more successive finite rotations in R^3 , around non concurrent axes, are equivalent, in infinitely many ways, to a pair of successive rotations.*

Perhaps one can find some pure geometric demonstration based for example on the fact that any rotation can be replaced by another rotation followed or preceded by a translation perpendicular to the rotation axis. Then by alternating cleverly these transformations and their inverses we might be able to reduce the p initial rotations to a pair of non concurrent rotations.

Why then should we study that subject with GA ? First because it is an instructive exercise, and second because then we are able to calculate explicitely the parameters of the final rotations at once from the given parameters of the initial rotations without proceeding step by step. That is much better for practical problems than the pure geometry.

Let the initial and final rotors be :

(13) $$R_i = \cos\frac{\theta_i}{2} + Iu_i \sin\frac{\theta_i}{2} \qquad B_j = \cos\frac{\varphi_j}{2} + Iv_j\, \sin\frac{\varphi_j}{2} \qquad i = 1, ..., p \qquad j = 1, 2$$

The anchor points are respectively a_i , b_j .

We will call R_0 the product R_1R_2 ....R_p which we evaluate by relations of type (5) and (6). R_0 is a rotor, but as we know by *theorem 1* , the composition of the rotations $\mathfrak{R}_1$,$\mathfrak{R}_2$, ...$\mathfrak{R}_p$ is not a rotation.

The reader will easily establish that, starting with any point x_0 , we get[1] :

(14) $$x_p = \tilde{R}_0\,(x_0 - a_1)R_0 + \tilde{R}_p...\tilde{R}_2(a_1 - a_2)R_2...R_p + ... + \tilde{R}_p(a_{p\perp 1} - a_p)R_p + a_p$$

(15) $$x_p = \tilde{B}_0\,(x_0 - b_1)B_0 + \tilde{B}_2(b_1 - b_2)B_2 + b_2$$

Then of course, because of x_0 , we must have :

(16) $$B_0 = B_1B_2 = R_0$$

1. We may note that (14) gives us immediately a decomposition of the global operator in a rotation $\tilde{R}_0\, x_0\, R_0$ and a translation w .

If we know all of B_2 then we know B_1 .

The above relations imply :

(17) $$\tilde{R}_0\,(b_2-b_1)R_0+\tilde{B}_2(b_1-b_2)B_2=$$
$$-\,b_2+\tilde{R}_0\,b_2R_0-\tilde{R}_0\,a_1R_0+\tilde{R}_p...\tilde{R}_2(a_1-a_2)R_2...R_p+....+\tilde{R}_p(a_{p\perp 1}-a_p)R_p+a_p=$$
$$-\,b_2+\tilde{R}_0\,b_2R_0+w$$

where w is a vector independent of b_2 and B_2 .

The above relation should allow us to evaluate (b_1-b_2) and b_1 by :

(18) $$b_1=(b_1-b_2)+b_2$$

We note that the second member of (17) depends on b_2 , but not at all on B_2 , that is on φ_2 and v_2 . We may hope to be able , by adjusting B_2 , to obtain admissible solutions for (b_1-b_2) without constraining b_2 . Now we try to solve a vector equation :

(19) $$-\,\tilde{R}_0\;y\,R_0+y+\tilde{B}_2\,y\,B_2-y=x$$

where $y=(b_1-b_2)$ and $x=-\,b_2+\tilde{R}_0\,b_2R_0+w$.

It is interesting to split y as follows :

(20) $$y=y.u_0\,u_0+y\wedge u_0\,u_0 \qquad y=y.v_2\,v_0+y\wedge v_2\,v_2$$

to get :

(21) $$y\wedge u_0\,u_0\,(1-R_0^2)+y\wedge v_2\,v_2(B_2^2-1)=x$$

Let us note that :

(22) $$(1-R_0^2)=(1-\cos\theta_0-Iu_0\sin\theta_0) \qquad (B_2^2-1)=(\cos\varphi_2-1+Iv_2\sin\varphi_2)$$

These expressions are rotors, dilated by scalar factors $2\sin(\theta_0/2)$ and $2\sin(\varphi_2/2)$. Thus in the first member of (21) we have the sum of two vectors, one lying in the Iu_0 plane, the other in the Iv_2 plane.

If a solution of equation (21) exists we should be able to write it as a sum of three independent well chosen vectors, that is :

(23) $$y=\alpha u_0+\beta v_2+\gamma Iu_0\wedge v_2$$

which we will use by taking advantage of all the possibilities at our disposal in GA. Unless of what happens in traditional vector calculus we let the basis vectors interact with the other factors in (21).

Without writing out all the details, that the reader might want to verify, we give following results :

$$y\wedge u_0\,u_0=\beta v_2\wedge u_0\,u_0+\gamma Iu_0\wedge v_2$$
$$y\wedge u_0\,u_0\,(1-R_0^2)=[\beta(\cos\theta_0-1)+\gamma\sin\theta_0]u_0\wedge v_2\,u_0+[\beta\sin\theta_0+\gamma(1-\cos\theta_0)]Iu_0\wedge v_2$$
$$y\wedge v_2\,v_2=\alpha u_0\wedge v_2\,v_2+\gamma Iu_0\wedge v_2$$
$$y\wedge v_2\,v_2(B_2^2-1)=[\alpha(\cos\varphi_2-1)-\gamma\sin\varphi_2]u_0\wedge v_2\,v_2+[\alpha\sin\varphi_2+\gamma(\cos\varphi_2-1)]Iu_0\wedge v_2$$

Thus we get :

(24) $$x=y\wedge u_0\,u_0\,(1-R_0^2)+y\wedge v_2\,v_2(B_2^2-1)=[\alpha(\cos\varphi_2-1)-\gamma\sin\varphi_2]u_0\wedge v_2\,v_2+$$
$$[\beta(\cos\theta_0-1)+\gamma\sin\theta_0]u_0\wedge v_2\,u_0+[\alpha\sin\varphi_2+\beta\sin\theta_0+\gamma(\cos\varphi_2-\cos\theta_0)]Iu_0\wedge v_2$$

We note that the basis vectors appearing in (24) are precisely, neglecting a scalar factor, the reciprocal vectors of u_0 , v_2 , $Iu_0 \wedge v_2$. Thus when multiplying (scalar product) (24) by those vectors we get immediately three linear equations in α, β, γ :

$$(25)\qquad \begin{aligned} &[(1-(u_0.v_2)^2][\alpha(\cos\varphi_2-1)+0\beta-\gamma\sin\varphi_2]=u_0.x \\ &[(u_0.v_2)^2-1][0\alpha+\beta(\cos\theta_0-1)+\gamma\sin\theta_0]=v_2.x \\ &[(1-(u_0.v_2)^2][\alpha\sin\varphi_2+\beta\sin\theta_0+\gamma(\cos\varphi_2-\cos\theta_0)]=(Iu_0\wedge v_2).x \end{aligned}$$

The determinant of that system appears to be nil :

$$(26)\qquad \begin{aligned} &(\cos\varphi_2-1)[(\cos\theta_0-1)(\cos\varphi_2-\cos\theta_0)-\sin^2\theta_0]-\sin\varphi_2[-\sin\varphi_2(\cos\theta_0-1)]= \\ &(\cos\theta_0-1)[(\cos\varphi_2-1)(\cos\varphi_2-\cos\theta_0+\cos\theta_0+1)+\sin^2\varphi_2]=0 \end{aligned}$$

But we would like the system to have an infinity of solutions. That is possible if and only if the determinant formed by replacing the last column in the previous matrix by the second member of (25) is also nil. So, developping by the first column we get :

$$(27)\qquad \begin{aligned} &(\cos\varphi_2-1)[(\cos\theta_0-1)(Iu_0\wedge v_2).x)+\sin\theta_0\, v_2.x]+\sin\varphi_2[-u_0.x(\cos\theta_0-1)]= \\ &-2\sin\tfrac{\varphi_2}{2}\{\sin\tfrac{\varphi_2}{2}[(\cos\theta_0-1)(Iu_0\wedge v_2).x)+\sin\theta_0\, v_2.x]+\cos\tfrac{\varphi_2}{2}(\cos\theta_0-1)u_0.x\}=0 \end{aligned}$$

Thus, as φ_2 should be different from 0 , we can always write :

$$(28)\qquad \mathrm{cotg}\tfrac{\varphi_2}{2}=[(\cos\theta_0-1)(Iu_0\wedge v_2).x)+\sin\theta_0\, v_2.x]/(1-\cos\theta_0)u_0.x$$

Remember that x depends linearly on b_2 but neither on φ_2 nor on v_2 .

Then we can, which was our goal, choose freely b_2 , then always fix φ_2 , v_2 so as to satisfy (28). We then calculate α, β, γ , perhaps by choosing $\gamma=0$ and solving (25), which gives us (b_1-b_2) .

Thus theorem 2 is proven, and we are able to determine numerically in infinitely many ways the pair $(\mathfrak{B}_1, \mathfrak{B}_2)$.

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