Vector Analysis with Geometric Algebra.

Thirty years ago C.T Tai (University of Michigan) wrote a book and three articles [1] [2] [3] concerning Vector Analysis. It was a thoroughful historic investigation and a successful try to eliminate all ambiguities. Successful but a little bit complicated. Let us try to show that GA resolves the problem in a much simpler way.

Some definitions.

First of all we consider an $\boldsymbol{R}^3$ space , which can be immediately extended to any $\boldsymbol{R}^n$ and to $\boldsymbol{M}^4$, that is to any flat space , with any system of curvilinear coordinates.

In GA we have the following self-explaining definitions :

1. $e_i(x) = \dfrac{\partial x}{\partial x^i} = \lim \dfrac{1}{\varepsilon}\,[x(\ldots, x^i + \varepsilon, \ldots.) - x]$

2. $e_i\,.e^j = \delta_i^j$ reciprocal vectors

3. $a.\nabla F(x) = \lim \dfrac{1}{\varepsilon}[F(x + \varepsilon a) - F(x)]$ and :

4. $e_i.\nabla F(x) = \lim \dfrac{1}{\varepsilon}[F(x + \varepsilon e_i) - F(x)] = \dfrac{\partial}{\partial x^i} F(x)$

It is obvious that an reciprocal frame exists and is unique. But in GA we are able write immediately :

5. $e_i = (-1)^{n-i}\, e^n \wedge e^{n-1} \wedge \ldots. \wedge \hat{e}^i \wedge \ldots. \wedge e^1\, IV$ where

   $\hat{e}^i =$ absent $I =$ pseudoscalar $V =$ elementary volume (scalar)

   $IV = e_1 \wedge e_2 \wedge \ldots\ldots \wedge e_n$ $I^2 = -1$

   It is a simple task verify that $e_i\,.e^j = \delta_i^j$

With (4) one can write :

6. $e_i.\nabla = \dfrac{\partial}{\partial x^i} = \partial_i$

7. $e_i.\nabla x^j = \dfrac{\partial x^j}{\partial x^i} = \delta_i^j$

Thus we are able to identify e^j with ∇x^j , that is with an important definition in modern differential calculus.

8. $e^j = \nabla x^j$

As a consequence we can write :

9. $\nabla = e^j \dfrac{\partial}{\partial x^j} = e^j \partial_j$

Last but not least we have to write a very important relation :

10. $\nabla \wedge e^j = \nabla \wedge (\nabla x^j) = 0$!!!

A first sight it seems obvious (the symmetry of the second differential) , but we will see later that it is more tricky.

What is most interesting is the fact that there exists no equivalent simple relation in standard vector and tensor analysis. If you add the fact that almost everyone of the mathematicians having written on the subject , either ignored or neglected the reciprocal vectors , you understand why were many approximations.

The curl in $\boldsymbol{R}^3$.

First of all we must write the relations *in any « natural system »* :

11. $$\nabla = e^j \frac{\partial}{\partial x^j}$$

12. $$a = e^i a_i = e_i a^i$$

Here as we want to use (10) it is obvious to choose $a = e^i a_i$. We get :

13. $$\nabla \wedge a = (\nabla \wedge e^i) a_i + \frac{\partial a_i}{\partial x^j} e^j \wedge e^i = \frac{\partial a_i}{\partial x^j} e^j \wedge e^i$$

The first member of (13) is of course coordinate-free , the second member can be modified to adopte orthonormal coordinates. One has to write :

14. $$\bar{a}_i = \lambda_i a^i = (\lambda_i)^{-1} a_i \quad \text{and} \quad \bar{e}^{\,j} = \lambda_j e^j = \bar{e}_j$$

where the λ are the local units of length.

We obtain :

15. $$\nabla \wedge a = \left[\frac{1}{\lambda_3 \lambda_2} \left(\frac{\partial \lambda_2 \bar{a}_2}{\partial x^3} - \frac{\partial \lambda_3 \bar{a}_3}{\partial x^2} \right) \bar{e}^{\,3} \wedge \bar{e}^{\,2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right]$$

We verify $\nabla \times a = -I \nabla \wedge a$.

We employ the same technique for all the other differentials. Thats all .... [4]

Gradient.

16. $$\nabla \phi = e^i \partial_i \phi = \bar{e}_i \frac{\partial \phi}{\lambda_i \partial x^i}$$

Divergence.

17. $$\nabla . a = (\nabla a^i) . e_i + a^i \nabla . e_i = \frac{\partial a^i}{\partial x^i} + a^i \nabla . e_i$$

18. $$\begin{aligned} \nabla . e_i &= \nabla . [\, (-1)^{n-i} e^n \wedge e^{n-1} \wedge \ldots . \wedge \hat{e}^i \wedge \ldots . \wedge e^1 I \,] V \\ &\quad + e^k . [\, (-1)^{n-i} e^n \wedge e^{n-1} \wedge \ldots . \wedge \hat{e}^i \wedge \ldots . \wedge e^1 I \,] \partial_k V \\ &= 0 + [\, e^n \wedge e^{n-1} \wedge \ldots . \wedge e^1 I \,] \partial_i V = V^{-1} \partial_i V \end{aligned}$$

19. $$\nabla . a = \frac{\partial a^i}{\partial x^i} + a^i \, \nabla . e_i = \frac{\partial a^i}{\partial x^i} + a^i V^{-1} \partial_i V = \frac{1}{V} \frac{\partial}{\partial x^i} (a^i V)$$

Then we get in orthonormal coordinates (with (14)) :

20. $$\nabla . a = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial x^1} \cdot (\lambda_2 \lambda_3 \bar{a}_1) + \frac{\partial}{\partial x^2} (\lambda_3 \lambda_1 \bar{a}_2) + \frac{\partial}{\partial x^3} \cdot (\lambda_1 \lambda_2 \bar{a}_3) \right]$$

The scalar laplacian.

One must be careful. In a simple form it appears as the divergence of a gradient:

21. $$\nabla^2 \phi = (\nabla . \nabla) \phi = \nabla . (\bar{e}^{\,i} \partial_i \phi) = \frac{\partial^2 \phi}{(\partial x^1)^2} + \frac{\partial^2 \phi}{(\partial x^2)^2} + \frac{\partial^2 \phi}{(\partial x^3)^2}$$

but in (19) you must replace a^i with $\partial_i \phi$ transformed in a contravariant quantity, that is $e^i . e^j \partial_j \phi$.

22. $$\nabla^2 \phi = \frac{1}{V} \frac{\partial}{\partial x^i} (V e^i . e^j \, \partial_j \phi)$$

In orthonormal coordinates we get :

23. $$\nabla^2 \phi = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial x^1} \left(\frac{\lambda_2 \lambda_3}{\lambda_1} \frac{\partial \phi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(\frac{\lambda_3 \lambda_1}{\lambda_2} \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(\frac{\lambda_1 \lambda_2}{\lambda_3} \frac{\partial \phi}{\partial x^3} \right) \right]$$

The vector laplacian.

It is defined by the obvious relation :

24. $$\nabla^2 a = (\nabla \nabla) a = \nabla (\nabla . a) + \nabla (\nabla \wedge a)$$

Additional remarks. [5]

For an $\boldsymbol{R}^n$ space (or $\boldsymbol{M}^4$) we know from tensor analysis :

25. $$\partial_k \, e_j = \Gamma^i_{kj} \, e_i \qquad \partial_k \, e^i = -\Gamma^i_{kj} \, e^j \qquad \Gamma^i_{kj} = \Gamma^i_{jk}$$

Thus we can write (10) :

26. $$\nabla \wedge e^j = e^k \wedge e^l \, \Gamma^j_{kl} = 0$$

In a Riemann space ($\mathcal{V}^n$) we must write :

27. $$\mathcal{D} \wedge e^j = e^k \wedge e^l \, \Gamma^j_{kl} = 0$$

$\mathcal{D}$ is the projection of ∇ on the manifold $\mathcal{V}^n$ (embedded in a $\boldsymbol{R}^{n+\cdots+}$).

G.Ringeisen january 2016

[1] A survey of the improper uses of ∇ in vector analysis. C.T.Tai 1994

[2] A historical study of vector analysis. C.T.Tai 1995

[3] Differential operators in vector analysis .... C.T.Tai

[4] Geometric algebra for Physicists Chris Doran Anthony Lasenby

[5] Spacetime Geometry with Geometric Calculus. David Hestenes